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COVARIANT EXPANSION OF A MODULAR FORM.

BY OLIVER E. GLENN.

A complete system of covariants of the total group G of linear transformations with integral coefficients modulo p , a prime number, is composed of*

$$L = x_1^p x_2 - x_1 x_2^p,$$

$$Q = (x_1^{p^2} x_2 - x_1 x_2^{p^2}) \div L = x_1^{p(p-1)} + x_1^{(p-1)(p-1)} x_2^{p-1} + \cdots + x_2^{p(p-1)}.$$

Consider a binary form, of order m , whose coefficients are independent variables:

$$f_m = (a_0, a_1, \dots, a_m) (x_1, x_2)^m = a_0 x_1^m + a_1 x_1^{m-1} x_2 + \cdots.$$

We propose to treat the problem of determining modular covariants ϕ_1, ϕ_2 of f_m such that the following congruence will hold identically in the a 's and in the x 's:

$$(1) \quad f_m \equiv Q\phi_1 + L\phi_2 \pmod{p}.$$

Regarding the forms ϕ_1, ϕ_2 in a relation like (1) to be general forms with undetermined coefficients, of respective orders $m - p^2 + p, m - p - 1$, it is evident that when $m = p^2$ the identity (1) implies $m + 1$ linear non-homogeneous equations between these coefficients. These linear equations are consistent, and hence ϕ_1, ϕ_2 are uniquely determined. In fact, in order to prove the consistency of this system of equations we have only to construct their matrix M , the elements of M being all 0, 1 or -1 . These elements are arranged in M by a simple law such that it is immediately evident that elementary transformations will reduce all elements to zero, excepting those in the principal diagonal, and that all in the diagonal will be $\equiv 1 \pmod{p}$. Hence if D is the determinant of M , $D \not\equiv 0 \pmod{p}$.

When, as in the case $m = p^2$ just mentioned, the quantics φ_1, φ_2 are uniquely determined, they are readily proved† to be formal covariants modulo p of f_m . Also, for a series of particular orders m , such that $m > p^2$, I have determined, non-uniquely, covariant pairs φ_1, φ_2 satis-

* Dickson, Transactions Amer. Math. Society, vol. 12 (1911), p. 75; and Madison Colloquium Lectures, 1913, p. 33.

† Cf. O. E. Glenn, Transactions Amer. Math. Society, vol. 18 (1917), p. 460.

fying (1). We shall give in tables computed covariants φ_1, φ_2 leading to an identity (1), for various cases in which $m \geq p^2$.

Whenever we have covariants* φ_1 and φ_2 of f_m for which (1) is an identity, we shall call (1) a covariant expansion, and φ_1 a principal covariant of f_m ; its seminvariant leader being a_0 . In general the covariant φ_1 leading to an expansion (1) is not unique. But any two principal covariants are linearly dependent modulo L ; indeed they all possess their real roots in common.† For, if φ_1, φ_1' are two such covariants their difference $\varphi_1' - \varphi_1$, being a covariant divisible by x_2 , contains the factor L , since the real points on the modular line are conjugate under G . Thus in the expansion corresponding to φ_1' , viz.,

$$(2) \quad f_m \equiv Q\varphi_1' + L\varphi_2' \pmod{p},$$

we may substitute

$$(3) \quad \varphi_1' \equiv \varphi_1 + L\psi \pmod{p},$$

where ψ is a formal covariant mod p of order $m - p^2 - 1$; and so (2) becomes

$$f_m \equiv Q\varphi_1 + L(Q\psi + \varphi_2') \pmod{p},$$

and the principal covariant in this expansion is φ_1 . The latter expansion is identical with (1); for, φ_2 may always be determined by dividing $f_m - Q\varphi_1$ by L , and the quotient modulo p is unique. Hence the covariant expansions are all transformable into a fixed one by congruences of type (3).

In case m is sufficiently large, φ_1 and φ_2 will be of order $\geq p^2$, and then, under restrictions similar to those described above for covariant expansions of f_m , φ_1, φ_2 may themselves be developed in covariant expansions; also, if only one of the covariants φ_1, φ_2 is of order $\geq p^2$, this one may be developed according to any expansion (1) known to exist for its order. Thus we arrive at a covariant expansion of f_m more explicit than (1), viz.,

$$(4) \quad f_m \equiv Q^\alpha\varphi_1 + Q^{p-1}L\chi + \cdots + QL^{r-1}\psi + L^s\omega \pmod{p},$$

in which the coefficient forms $\varphi_1, \chi, \dots, \psi, \omega$ are covariants of f_m of orders $< p^2$; the principal covariant being φ_1 led by a_0 . This expansion is not unique.

The existence and explicit form of the principal formal covariants modulo 2 for the general order m were demonstrated by the present writer

* The covariancy of φ_1 evidently implies that of φ_2 .

† Since the roots of $L \equiv 0 \pmod{p}$ are the real residues mod p , and those of $Q \equiv 0$ are the Galoisian imaginaries which are roots of irreducible quadratic congruences mod p , the real roots of $f_m \equiv 0 \pmod{p}$ are also roots of $\varphi_1 \equiv 0$, and such imaginary roots of $f_m \equiv 0$ are also roots of $\varphi_2 \equiv 0 \pmod{p}$ (cf. Dickson, Madison Colloquium Lectures, p. 37).

in the Transactions of the American Mathematical Society, volume 17 (p. 545). For m even, φ_1 is the covariant

$$K_2 = a_0x_1^2 + (a_1 + \cdots + a_{m-1})x_1x_2 + a_mx_2^2,$$

while for m odd it was proved that φ_1 is a cubic covariant

$$K_{m3} = a_0x_1^3 + I_1x_1^2x_2 + I_2x_1x_2^2 + a_mx_2^3,$$

where I_1 , I_2 are of somewhat complex general structure but such that $I_1 + I_2$ is congruent to the invariant $a_1 + \cdots + a_{m-1}$. Illustrations of K_2 and K_{m3} are contained in the lists below, showing expansions modulo 2 for forms of the first eleven orders, and of the forms of orders 9, 10, 11 when the modulus is 3. The existence of the principal covariants in the latter three cases was established by methods similar to those employed for the modulus 2, but for $m = 11$, in order to exhibit a complete set of principal covariants in one formula, by retaining a parameter λ in their coefficients, φ_1 was assumed in its general form and the conditions for its invariancy under the induced group were imposed. This method required the solution of a set of linear congruences in twenty unknowns, and was not brief, but, once these covariants are found, the direct verification of their covariancy is easy.

Tables.

We employ the notation $(hijk\cdots)$ for the sum

$$a_h + a_i + a_j + a_k + \cdots$$

$$p = 2, \quad m = 1, 2, 3.$$

The forms f_1, f_2, f_3 are irreducible.

$$p = 2, \quad m = 2^2.$$

$$f_4 \equiv QK_2 + L(K_1 + C_{101})(\text{mod } 2).$$

$$K_2 \equiv (0)x_1^2 + (123)x_1x_2 + (4)x_2^2,$$

$$K_1 \equiv (0123)x_1 + (1234)x_2,$$

$$C_{101} \equiv (1)x_1 + (3)x_2.$$

$$p = 2, \quad m = 5.$$

$$f_5 \equiv QK_{53} + L(K_2 + C_{102})(\text{mod } 2).$$

$$K_{53} \equiv (0)x_1^3 + (12)x_1^2x_2 + (34)x_1x_2^2 + (5)x_2^3,$$

$$K_2 \equiv (0)x_1^2 + (1234)x_1x_2 + (5)x_2^2,$$

$$C_{102} \equiv (2)x_1^2 + (3)x_2^2.$$

$$p = 2, \quad m = 6.$$

$$f_6 \equiv Q^2 K_2 + LK_3 \pmod{2}.$$

$$K_2 \equiv (0)x_1^2 + (12345)x_1x_2 + (6)x_2^2,$$

$$K_3 \equiv (2345)x_1^3 + (03456)x_1^2x_2 + (01236)x_1x_2^2 + (1234)x_2^3.$$

$$p = 2, \quad m = 7.$$

$$f_7 \equiv Q^2 K_{73} + LQM + L^2 J \pmod{2}.$$

$$K_{73} \equiv (0)x_1^3 + (124)x_1^2x_2 + (356)x_1x_2^2 + (7)x_2^3,$$

$$M \equiv (24)x_1^2 + (35)x_2^2,$$

$$J \equiv (02356)x_1 + (12457)x_2.$$

$$p = 2, \quad m = 8.$$

$$f_8 \equiv Q^3 K_2 + QLK_{53}' + L^2(K_2' + C_{102}') \pmod{2}.$$

$$K_2 \equiv (0)x_1^2 + (1234567)x_1x_2 + (8)x_2^2,$$

$$K_{53}' \equiv (0234567)x_1^3 + (038)x_1^2x_2 + (058)x_1x_2^2 + (1234568)x_2^3,$$

$$K_2' \equiv (0234567)x_1^2 + (35)x_1x_2 + (1234568)x_2^2,$$

$$C_{102}' \equiv (123)x_1^2 + (567)x_2^2.$$

$$p = 2, \quad m = 9.$$

$$f_9 \equiv Q^3 K_{93} + Q^2 LK_2' + L^2 K_3' \pmod{2}.$$

$$K_{93} \equiv (0)x_1^3 + (1234)x_1^2x_2 + (5678)x_1x_2^2 + (9)x_2^3,$$

$$K_2' \equiv (0234)x_1^2 + (124578)x_1x_2 + (5679)x_2^2,$$

$$K_3' \equiv (046)x_1^3 + (157)x_1^2x_2 + (248)x_1x_2^2 + (359)x_2^3.$$

$$p = 2, \quad m = 10.$$

$$f_{10} \equiv Q^4 K_2 + Q^2 LK_{73}' + QL^2 M' + L^3 J' \pmod{2}.$$

$$K_2 \equiv (0)x_1^2 + (123456789)x_1x_2 + (10)x_2^2,$$

$$K_{73}' \equiv (23456789)x_1^3 + (0124510)x_1^2x_2 + (0568910)x_1x_2^2 + (12345678)x_2^3,$$

$$M' \equiv (01236789)x_1^2 + (123478910)x_2^2,$$

$$J' \equiv (0235689)x_1 + (12457810)x_2.$$

$$p = 2, \quad m = 11.$$

$$f_{11} \equiv Q^4 K_{113} + Q^3 LK_2' + QL^2 K_{53}'' + L^3(K_2'' + C_{102}'') \pmod{2}.$$

$$K_{113} \equiv (0)x_1^3 + (12458)x_1^2x_2 + (367910)x_1x_2^2 + (11)x_2^3,$$

$$K_2' \equiv (2458)x_1^2 + (56)x_1x_2 + (3679)x_2^2,$$

$$K_{53}'' \equiv (2357910)x_1^3 + (02341011)x_1^2x_2 + (0178911)x_1x_2^2 + (124689)x_2^3,$$

$$K_2'' \equiv (2357910)x_1^2 + (123478910)x_1x_2 + (124689)x_2^2,$$

$$C_{102}'' \equiv (035678910)x_1^2 + (123456811)x_2^2.$$

$$p = 3, \quad m = 3^2.$$

$$f_9 \equiv Q\varphi_1 + L\varphi_2 \pmod{3}.$$

$$\varphi_1 \equiv (0)x_1^3 + (1357)x_1^2x_2 + (2468)x_1x_2^2 + (9)x_2^3,$$

$$\begin{aligned} \varphi_2 \equiv & 2(a_3 + a_5 + a_7)x_1^5 + 2(a_0 + a_4 + a_6 + a_8)x_1^4x_2 \\ & + (2a_1 + 2a_3 + a_5 + a_7 + 2a_9)x_1^3x_2^2 \\ & + (a_0 + 2a_2 + 2a_4 + a_6 + a_8)x_1^2x_2^3 \\ & + (a_1 + a_3 + a_5 + a_9)x_1x_2^4 + (a_2 + a_4 + a_6)x_2^5. \end{aligned}$$

$$p = 3, \quad m = 10.$$

$$f_{10} \equiv Q\varphi_1 + L\varphi_2 \pmod{3}.$$

$$\begin{aligned} \varphi_1 \equiv & a_0x_1^4 + (a_1 + a_3 + 2a_5)x_1^3x_2 + (a_2 + a_4 + a_6 + a_8)x_1^2x_2^2 \\ & + (2a_5 + a_7 + a_9)x_1x_2^3 + a_{10}x_2^4, \end{aligned}$$

$$\begin{aligned} \varphi_2 \equiv & (2a_3 + a_5)x_1^6 + 2(a_0 + a_4 + a_6 + a_8)x_1^5x_2 \\ & + 2(a_1 + a_3 + a_7 + a_9)x_1^4x_2^2 \\ & + (a_0 + 2a_2 + 2a_4 + a_6 + a_8 + 2a_{10})x_1^3x_2^3 \\ & + (a_1 + a_3 + a_7 + a_9)x_1^2x_2^4 + (a_2 + a_4 + a_6 + a_{10})x_1x_2^5 \\ & + (2a_5 + a_7)x_2^6. \end{aligned}$$

$$p = 3, \quad m = 11.$$

The abbreviations employed are as follows: λ is any least residue modulo 3; S represents the seminvariant $a_1 + a_3 + a_5 + a_7 + a_9$, and T the anti-seminvariant $a_2 + a_4 + a_6 + a_8 + a_{10}$.

$$f_{11} \equiv Q\varphi_1 + L\varphi_2 \pmod{3}.$$

$$\begin{aligned}\varphi_1 \equiv & a_0x_1^5 + (a_1 + a_3 + \lambda S)x_1^4x_2 + (a_2 + a_4 + a_6 - \lambda T)x_1^3x_2^2 \\ & + (a_5 + a_7 + a_9 - \lambda S)x_1^2x_2^3 + (a_8 + a_{10} + \lambda T)x_1x_2^4 + a_{11}x_2^5, \\ \varphi_2 \equiv & (-a_3 - \lambda S)x_1^7 + (-a_0 - a_4 - a_6 + \lambda T)x_1^6x_2 - (\lambda + 1)Sx_1^5x_2^2 \\ & + (a_0 - a_6 + (\lambda + 2)T)x_1^4x_2^3 + (a_5 - a_{11} - (\lambda + 2)S)x_1^3x_2^4 \\ & + (\lambda + 1)Tx_1^2x_2^5 + (a_5 + a_7 + a_{11} - \lambda S)x_1x_2^6 + (a_8 + \lambda T)x_2^7.\end{aligned}$$

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